*J. Fluid Mech.* (1983), *vol.* 137, *pp.* 393–407
*Printed in Great Britain*



# Peristaltic pumping in water waves

By M. S. LONGUET-HIGGINS
Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge, and Institute of Oceanographic Sciences, Wormley, Surrey

(Received 21 January 1983)

In this paper we calculate the streaming induced by gravity waves passing over a thin fluid layer, one side of which is rigid while the other is a flexible, inextensible membrane. The problem is relevant to some recent laboratory experiments by Allison (1983) on the pumping action of water waves.

On the assumption that the flow is laminar and that the lateral displacement $b$ of the membrane is small compared with the thickness $\Delta$ of the fluid layer, we calculate the velocity profile of the streaming $U$ within the layer. This depends on the ratio $\Delta/\delta$, where $\delta$ is the thickness of the Stokes layers at the upper and lower boundaries. When $0 < \Delta/\delta < 6$ the boundary layers interact strongly and the velocity profile is unimodal. At large values of $\Delta/\delta$ the profile of $U$ exhibits thin 'jets' near the boundaries.

The calculated drift velocities agree as regards order of magnitude with those observed. However, the pressure gradients observed were larger than those calculated, due possibly to turbulence, but probably also to finite-amplitude and end-effects.

The theory given here can be considered as an extension of the theory of peristaltic pumping to flows at higher Reynolds number.

## 1. Introduction

In a recent experiment to extract power from the mass transport in water waves, Allison (1983) laid a flexible bag, 6 m long and 0.5 m wide, on the floor of a wave basin, with the longer side in the direction of wave propagation. The two ends of the bag were connected externally by a rigid pipe. In the presence of gravity waves of 0.8–2.5 s period it was found that mean circulation of fluid took place down-wave through the bag, returning via the pipe. If the pipe was constricted, a mean head of 1–2 cm of water was built up.

The aim of the present note is to analyse the fluid mechanics of this effect. We shall show that the pumping action is due very largely to viscosity, being similar to that occurring in organic tubes (Jaffrin & Shapiro 1971).

In any oscillating flow, the importance of viscosity in inducing a steady streaming close to a rigid or flexible boundary has been known ever since Rayleigh (1884) analysed the currents induced by standing oscillations in air or water. On the other hand for progressive motions, the streaming tangential to a membrane or solid boundary was first evaluated by Longuet-Higgins (1953). His analysis showed that just outside the Stokes layer, of thickness $\delta = (2\nu/\sigma)^{\frac{1}{2}}$, where $\nu$ is the kinematic viscosity and $\sigma$ the radian frequency of oscillation, the tangential streaming velocity tended to the finite value

$$U = \frac{5}{4}\frac{q^2}{c}. \tag{1.1}$$

Here $q$ denotes the amplitude of the first-order oscillatory velocity relative to the boundary, and $c$ is the phase speed. Particularly interesting is the fact that the limiting value (1.1) is independent of $\nu$ and so remains non-negligible even when the Stokes layer itself is quite thin. This forwards streaming or jet near the bottom had been first noticed in water waves by Bagnold (1947), and was later confirmed by Russell & Osorio (1956) and many others.

It is a forwards streaming of this type which we suggest controls the flow within the flexible bag in Allison's experiment. A preliminary analysis is given below in §2, assuming the flow to be laminar and the separation $\Delta$ between upper and lower surfaces of the bag to be large compared with the thickness $\delta$ of the Stokes layers. In §§3 and 4 we extend the analysis to the situation of turbulent flow, and to when $\Delta/\delta$ is not necessarily large. The theory gives a reasonable agreement with Allison's experiment. Further discussion follows in §§6 and 7.

## 2. A laminar model: $\delta \ll \Delta$

The situation is idealized as in figure 1 (*a*). Homogeneous, incompressible fluid is contained between a rigid plane $z = 0$ and a flexible membrane at a mean distance $\Delta$ above the bottom. The vertical oscillations in the membrane are of amplitude $b$ and travel to the right with speed $c$, so that the equation of the membrane is

$$z = \Delta + b\cos(kx - \sigma t), \tag{2.1}$$

where $x$ and $z$ are horizontal and vertical coordinates, $k$ is the wavenumber, and $\sigma/k = c$. We shall assume at first that

$$k\Delta \ll 1, \tag{2.2}$$

that is, the wavelength is long compared with the thickness of the fluid layer, and

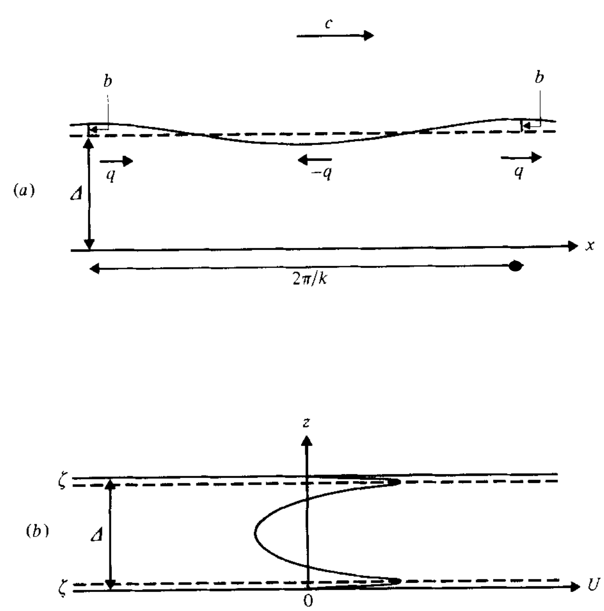


FIGURE 1. (*a*) A two-dimensional model of the flexible bag. (*b*) A typical profile of the streaming velocity $U$.

also

$$b/\Delta \ll 1, \tag{2.3}$$

that is, the vertical displacements are only a small fraction of the layer thickness; this latter restriction will be modified later. The motion will be treated at first as two-dimensional and independent of the $y$-coordinate.

The vertical velocity $w$ vanishes on the bottom and, when $z = \Delta$, $w$ is given by $b\sigma \sin(kx-\sigma t)$ to first order in $bk$. Hence in general

$$w = \frac{z}{\Delta} b\sigma \sin(kx-\sigma t) \tag{2.4}$$

everywhere outside the boundary-layers.

To begin with we shall assume

$$\delta = (2\nu/\sigma)^{\frac{1}{2}} \ll \tfrac{1}{2}\Delta, \tag{2.5}$$

so that outside the boundary-layer the horizontal velocity is given by

$$u = \frac{b\sigma}{k\Delta}\cos(kx-\sigma t) \tag{2.6}$$

to first order in $bk$. To this order the tangential velocity of the membrane is negligible, so that in (1.1) the amplitude $q$ of the oscillatory velocity relative to the boundary is

$$q = \frac{b\sigma}{k\Delta} = \frac{bc}{\Delta} \tag{2.7}$$

at both upper and lower boundaries.

To evaluate the mean flow we note that, if there were no mean horizontal gradient in the pressure, the streaming velocity would be uniform and equal to (1.1) everywhere in the interior. In the presence of a mean pressure gradient $\varpi$ we have to solve the equation

$$\nu \frac{\partial^2 \bar{u}}{\partial z^2} = \frac{\varpi}{\rho} + \overline{u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}} \tag{2.8}$$

(see Longuet-Higgins 1953), where $\bar{u}$ is the time-mean velocity at a fixed point. This is related to the streaming velocity $U$ by

$$U = \bar{u} + \overline{\int u\,\mathrm{d}t \frac{\partial u}{\partial x}} + \overline{\int w\,\mathrm{d}t\frac{\partial u}{\partial z}}, \tag{2.9}$$

the last two terms representing the Stokes drift. In view of (2.4) and (2.6), this reduces to

$$U = \bar{u} + \tfrac{1}{2}q^2/c. \tag{2.10}$$

Since the second term on the right of (2.10) is independent of $z$, we have for $U$ in the interior the same equation as for $\bar{u}$, namely

$$\frac{\partial^2 U}{\partial z^2} = \frac{\varpi}{\rho\nu} \tag{2.11}$$

with boundary conditions

$$U = \frac{5}{4}\frac{q^2}{c} \quad \text{when} \quad z \doteqdot 0, \Delta \tag{2.12}$$

(see figure 1*b*). The use of $\doteqdot$ signifies that the condition has actually to be satisfied just outside each boundary-layer, but since $\delta \ll \Delta$ the difference is negligible. In addition, to second order in $bk$ the boundary condition can be taken as satisfied at the mean vertical position of the membrane, that is at $z = \Delta$ and not on (2.1).

The solution to the problem (2.11), (2.12) is clearly

$$U = \frac{5}{4}\frac{q^2}{c} + \frac{1}{2}\frac{\varpi}{\rho\nu}z(z-\Delta). \tag{2.13}$$

In other words, we add to the uniform velocity (1.1) a parabolic flow, symmetric about the central line, as in figure 1 ($b$).

Corresponding to (2.13), the total volume flux $M$ is given by

$$M = \frac{5}{4}\frac{q^2\Delta}{c} - \frac{1}{12}\frac{\varpi\Delta^3}{\rho\nu}. \tag{2.14}$$

If $\varpi$ vanishes, the mean velocity is simply

$$\bar{U} = \frac{5}{4}\frac{q^2}{c}, \tag{2.15}$$

while, if the flow is blocked so that $M$ vanishes, we induce a pressure head

$$[p] = \varpi L = 15\frac{\rho\nu L}{\Delta^2}\frac{q^2}{c}, \tag{2.16}$$

where $L$ is the total length of the flexible tube. In general the power $P$ available per unit width of the tube is

$$P = \varpi LM. \tag{2.17}$$

From (2.14) this is a maximum when

$$\varpi = \frac{15}{2}\frac{\rho\nu}{\Delta^2}\frac{q^2}{c}, \tag{2.18}$$

so the maximum available power is given by

$$P_{\max} = \frac{75}{16}\frac{\rho\nu L}{\Delta}\frac{q^4}{c^2}. \tag{2.19}$$

In all these expressions $q^2/c$ may be replaced by $(b/\Delta)^2 c$, where $2b$ is the overall vertical displacement of the membrane.

## 3. Comparison with observation

In Allison's experiment (1983, figure 6) the wave period $T = 2\pi/\sigma$ ranged from 0.8 to 2.5 s, in water of constant depth $h = 30$ cm. The wave height $2a$ was greatest at about 5.5 cm for waves of period about 1.2 s.

The amplitude $b$ of the vertical displacement of the membrane was however measured as 1.0 cm for waves of 1.1 s period. This compares with a notional amplitude

$$b' = a\frac{\sinh k\Delta}{\sinh kh} = 0.27 \text{ cm} \tag{3.1}$$

of vertical fluid motion at a height $\Delta$ above the bottom, in the absence of the flexible bag. From Allison's figure 2 we take $\Delta = 4.0$ cm. Since $b > b'$ it appears that the presence of the membrane does affect the waves above the bag, increasing their amplitude, possibly by wave refraction.

For the waves of period 1.1 s we have from the linear dispersion relation

$$kh \tanh kh = \sigma^2 h/g = 1.00, \tag{3.2}$$

so

$$kh = 1.20 \tag{3.3}$$

and the wavelength $2\pi/k$ is about 1 m or three times the depth $h$. The phase speed $c$ is 143 cm/s, and if we assume the ratio $b/\Delta$ in (2.7) to be constant despite variations in $\Delta$ across the width of the bag, then $q$ is constant at $0.25c$ or about 36 cm/s. The mean-flow velocity (2.15) in the absence of an external pressure gradient would be 11.2 cm/s, in apparently good agreement with the measured velocity 13 cm/s at this wave period; see Allison's figure 8.

However two adjustments must be made. The cross-section $\Omega$ of the external pipe, diameter $d = 9.0$ cm, at the point of measured velocity, was

$$\Omega = \tfrac{1}{4}\pi d^2 = 64 \text{ cm}^2 \tag{3.4}$$

compared with the cross-section of the flexible bag, of width $W = 50$ cm. Assuming the upper surface of the bag to be parabolic, the cross-sectional area $\Omega'$ would be

$$\Omega' = \tfrac{2}{3}W\Delta = 133 \text{ cm}^2, \tag{3.5}$$

so that the theoretical flow velocity should be multiplied by a factor $\Omega'/\Omega = 2.1$. On the other hand, the external resistance to the flow, both from the turbine and from a pipe of non-uniform cross-section, would tend to reduce the flow, so that the observed flow is not necessarily in disagreement with the simple theory.

Consider the dependence of the flow velocity upon the wave period $T$. The effects of refraction, etc. being difficult to assess, we shall assume roughly that the amplitude $b$ of the vertical displacement of the bag varied simply in proportion to the theoretical displacement $b'$. Carrying through the same calculation as above for $T = 1.1$ s, we arrive at the numbers shown in table 1. The velocity

$$V_{\max} = \frac{\Omega'}{\Omega}\frac{5}{4}\frac{q^2}{c} \tag{3.6}$$

shown on the right of table 1 will be seen to behave qualitatively in a similar way to the measured velocity, with a maximum at around $T = 1.2$ s instead of 1.1 s.

Consider now the maximum pressure head $[p]$ as given by (2.16). Clearly this depends critically on the thickness $\Delta$ of the fluid layer. Since in the experiment $\Delta$ was non-uniform over the width of the bag, being smaller near the two sides, (2.16) suggests there may have been some lateral circulation within the bag, in contrast with

| $T$ (s) | $\dfrac{\sigma^2 h}{g}$ | $kh$ | $c$ (cm/s) | $2a$ (cm) | $q$ (cm/s) | $\dfrac{q^2}{c}$ (cm/s) | $V_{\max}$ (cm/s) | $[p]$ (dyn/cm$^2$) |
|---|---|---|---|---|---|---|---|---|
| 0.8 | 1.886 | 1.962 | 120 | 3.0 | 13 | 1.3 | 3.4 | 21 |
| 0.9 | 1.490 | 1.613 | 160 | 4.5 | 24 | 3.2 | 8.3 | 53 |
| 1.0 | 1.207 | 1.373 | 137 | 5.0 | 31 | 6.9 | 17.9 | 113 |
| 1.1 | 0.998 | 1.198 | 143 | 5.0 | 35 | 8.9 | 23.5 | 143 |
| 1.2 | 0.838 | 1.064 | 148 | 5.5 | 42 | 11.8 | 30.6 | 194 |
| 1.3 | 0.714 | 0.960 | 151 | 5.0 | 40 | 10.6 | 27.5 | 174 |
| 1.4 | 0.616 | 0.875 | 154 | 4.2 | 35 | 8.0 | 20.8 | 132 |
| 1.5 | 0.537 | 0.805 | 156 | 4.2 | 36 | 8.4 | 17.7 | 138 |
| 2.0 | 0.302 | 0.579 | 163 | 3.5 | 33 | 6.8 | 20.8 | 112 |
| 2.5 | 0.193 | 0.454 | 166 | 2.5 | 25 | 3.7 | 9.6 | 61 |

TABLE 1. Wave parameters in figure 6 of Allison (1983)

our two-dimensional model. To estimate the actual mean pressure gradient it is perhaps reasonable to replace $\Delta$ by its mean value $\frac{2}{3}\Delta_0$, where $\Delta_0 = 4.0$ cm, the value at the centre of the cross-section. Then on taking $\nu = 0.013$ cm$^2$/s we find for $[p]$ the numerical values given in the right-hand column of table 1. Clearly these are less than the observed mean pressures in Allison's figure 8 by an order of magnitude.

Inspection of table 1 suggests one possible reason for the discrepancy, for the Reynolds numbers

$$R = q\Delta/\nu \tag{3.7}$$

are of order $10^3$, implying that the flow within the flexible bag is possibly turbulent.

Now a formal extension of the theory of oscillatory boundary-layers to turbulent flow was given by Longuet-Higgins (1956), who showed that if the molecular viscosity $\nu$ were replaced by a coefficient of kinematic viscosity $N$ which was a function only of the mean distance $\bar{z}$ of a particle from the boundary ($N$ being constant following a particle), then in a progressive wave motion, though the details of the boundary layer depended on the form of $N(\bar{z})$, the limiting drift velocity $U$ outside the layer was unaffected. That is to say (1.1) remained valid. Indeed, some such result was necessary to explain the observations by Russell & Osorio (1956).

In a turbulent flow the complete velocity profile (corresponding to (2.13) in the laminar conditions) must depend on the function $N(\bar{z})$. The observational evidence, combined with equation (2.16), suggests that in order of magnitude $N$ should be about $10\nu$. The result of increasing the effective viscosity by this amount would be to multiply the thickness $\delta$ of the boundary layer by a factor $(N/\nu)^{\frac{1}{2}}$, or about 3. For waves of 1 s period, for instance, $\delta$ would be increased from 0.064 cm to about 0.2 cm. As this is beginning to be comparable to the half-thickness of the layer within the bag, there may in fact be some interaction between the upper and lower boundary layers, leading to a reduction in the net mean flow. We investigate this effect in §4.

## 4. Boundary-layer interactions: $\delta/\Delta = O(1)$

We now extend and generalize the analysis of §2 to a situation when the thickness of the boundary-layer is no longer small compared with the thickness of the fluid layer in the flexible bag. Thus we assume $\delta/\Delta = O(1)$, where $\delta = (2\nu/\sigma)^{\frac{1}{2}}$. However, the restrictions

$$k\Delta \ll 1, \quad b/\Delta \ll 1 \tag{4.1}$$

will be retained. The first of these implies that the wavelength is long compared with the thickness $\Delta$ of the layer, so that $\partial/\partial x \ll \partial/\partial z$ in general. The second implies that the stream function may be expanded in the form

$$\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + \ldots, \tag{4.2}$$

where $\epsilon$ is a small parameter of order $b/\Delta$, and we may use the equations for the mass transport and mean flow developed by Longuet-Higgins (1953).

Thus for the first-order flow $\psi_1$ we have the differential equation

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2\psi_1 = 0, \tag{4.3}$$

in which $\nabla^2$ may be approximated by $\partial^2/\partial z^2$. The boundary conditions are that

$$\psi_{1x} = 0, \quad \psi_{1z} = 0 \quad \text{when} \quad z = 0, \tag{4.4}$$

$$\epsilon\psi_{1x} = -w, \quad \psi_{1z} = 0 \quad \text{when} \quad z = \Delta, \tag{4.5}$$

where $w$ denotes the vertical velocity imposed at the upper membrane. For a progressive wave this is given by (2.4). Using complex notation, so $\psi_1 \propto e^{i(kx-\sigma t)}$, we have

$$\psi_{1zzt} - \nu\psi_{1zzzz} = 0 \tag{4.6}$$

with

$$\psi_1 = 0, \quad \psi_{1z} = 0 \quad \text{when} \quad z = 0, \tag{4.7}$$

$$\psi_1 = c\Delta\, e^{i(kx-\sigma t)}, \quad \psi_{1z} = 0 \quad \text{when} \quad z = \Delta. \tag{4.8}$$

The solution of these equations is

$$\psi_1 = [A(\cosh\alpha z - 1) + B(\sinh\alpha z - \alpha z)]\, e^{i(kx-\sigma t)}, \tag{4.9}$$

where

$$\alpha = \left(\frac{\sigma}{i\nu}\right)^{\frac{1}{2}} = \frac{1-i}{\delta}, \tag{4.10}$$

and $A$ and $B$ are constants, satisfying

$$\left.\begin{aligned} A(\cosh\alpha\Delta - 1) + B(\sinh\alpha\Delta - \alpha\Delta) &= c\Delta, \\ A\sinh\alpha\Delta + B(\cosh\alpha\Delta - 1) &= 0. \end{aligned}\right\} \tag{4.11}$$

For the stream function $\epsilon^2\overline{\psi}_2$ of the mean motion at a fixed point in the fluid, we have from the momentum equations

$$\overline{\psi_{1z}\psi_{1xz}} - \overline{\psi_{1x}\psi_{1zz}} = -\frac{\varpi}{\rho\epsilon^2} + \nu\overline{\psi}_{2zzz}. \tag{4.12}$$

$\varpi$ is the mean horizontal pressure gradient, $\rho$ is the density, and an overbar denotes the mean value with respect to time. The terms on the left represent the convected momentum.

Since the motion is periodic in the $x$-direction, the first term $\overline{\psi_{1z}\psi_{1xz}}$ vanishes identically. In the second term, since the motion is progressive, we may replace $\partial/\partial x$ by $-c^{-1}\partial/\partial t$, and then use the property that if $F$ and $G$ are any two periodic quantities

$$\overline{F_t G + F G_t} = 0. \tag{4.13}$$

Hence

$$\overline{\psi_{1x}\psi_{1zz}} = -\frac{1}{c}\overline{\psi_{1t}\psi_{1zz}} = \frac{1}{c}\overline{\psi_1\psi_{1zzt}}. \tag{4.14}$$

On substituting for $\psi_{1zzt}$ from (4.6) we obtain for $\overline{\psi}_2$ the equation

$$\overline{\psi}_{2zzz} = -\frac{1}{c}\overline{\psi_1\psi_{1zzzz}} + \frac{\varpi}{\rho\nu\epsilon^2}. \tag{4.15}$$

To obtain the mass-transport velocity we note that the stream function $\epsilon^2\Psi$ for the mass transport is related to $\overline{\psi}_2$ by

$$\Psi = \overline{\psi}_2 + \overline{\int \psi_{1z}\, dt\, \psi_{1x}} \tag{4.16}$$

in general (see Longuet-Higgins 1953, equation (36)). The second group of terms on the right corresponds to the Stokes drift. For progressive motion, on replacing $\psi_{1x}$ by $-c^{-1}\psi_{1t}$ and using (4.13) we have

$$\overline{\int \psi_{1z}\, dt\, \psi_{1x}} = -\frac{1}{c}\overline{\int \psi_{1z}\, dt\, \psi_{1t}} = \frac{1}{c}\overline{\psi_{1z}\psi_1}, \tag{4.17}$$

so that (4.14) can be written

$$\Psi = \overline{\psi}_2 + \frac{1}{2c}(\overline{\psi_1^2})_z. \tag{4.18}$$

From (4.15) and (4.18) we obtain for $\Psi$ the equation

$$\Psi_{zzz} = \frac{1}{2c}(\overline{\psi_1^2})_{zzzz} - \frac{1}{c}\overline{\psi_1\psi_{1zzzz}} + \frac{\varpi}{\rho\nu\epsilon^2}. \tag{4.19}$$

This can be further simplified. For by Leibnitz's theorem the first two terms on the right can be written

$$\frac{1}{c}(\overline{\psi_1\psi_{1zzzz}} + 4\overline{\psi_{1z}\psi_{1zzz}} + 3\overline{\psi_{1zz}^2}) - \frac{1}{c}\overline{\psi_1\psi_{1zzzz}} = \frac{1}{c}(\overline{\psi_{1z}\psi_{1zzz}} + \frac{3}{2}\frac{\partial^2}{\partial z^2}\overline{\psi_{1z}^2}). \tag{4.20}$$

But from (4.6) we have

$$\psi_{1zz} = \nu\int\psi_{1zzzz}\,\mathrm{d}t, \tag{4.21}$$

and so on integration with respect to $z$

$$\psi_{1z} = \nu\int\psi_{1zzz}\,\mathrm{d}t + Q, \tag{4.22}$$

where $Q$ is independent of $z$. In fact

$$Q = \frac{\nu}{\mathrm{i}\sigma}(\psi_{1zzz})_{z=0} = -\alpha B\,\mathrm{e}^{\mathrm{i}(kx-\sigma t)}. \tag{4.23}$$

Multiplying (4.22) by $\psi_{1zzz}$ and averaging, we have

$$\overline{\psi_{1z}\psi_{1zzz}} = \overline{Q\psi_{1zzz}}. \tag{4.24}$$

From (4.19), (4.20) and (4.24) we have altogether

$$\Psi_{zzz} = \frac{1}{c}\overline{Q\psi_{1zzz}} + \frac{3}{2c}(\overline{\psi_{1z}^2})_{zz} + \frac{\varpi}{\rho\nu\epsilon^2}. \tag{4.25}$$

To obtain the mass-transport velocity

$$U = \epsilon^2\Psi_z \tag{4.26}$$

we now need only integrate (4.25) twice with respect to $z$, subject to the two boundary conditions

$$U = 0 \quad \text{when} \quad z = 0 \quad \text{and} \quad z = \Delta. \tag{4.27}$$

Since $\psi_{1z}$ vanishes on both boundaries, we have immediately

$$U = \frac{\epsilon^2}{c}(\overline{Q\psi_{1z}} + \tfrac{3}{2}\overline{\psi_{1z}^2}) + \frac{\varpi}{2\rho\nu}z(z-\Delta). \tag{4.28}$$

The total volume flux $M$ is equal to $\epsilon^2[\Psi]_0^\Delta$, and so

$$M = \frac{\epsilon^2}{c}\left\{\overline{Q(\psi_1)_{z=\Delta}} + \frac{3}{2}\int_0^\Delta \overline{\psi_{1z}^2}\,\mathrm{d}z\right\} - \frac{\varpi\Delta^3}{12\rho\nu}. \tag{4.29}$$

In evaluating the time-averaged terms in (4.28) and (4.29) we may use the complex expressions (4.9) and (4.23) for $\psi_1$ and $Q$ together with the rule

$$\overline{FG} = \tfrac{1}{4}(FG^* + F^*G) = \tfrac{1}{2}\,\mathrm{Re}\,FG^*. \tag{4.30}$$

## 5. Discussion

Consider first the case when $\Delta \gg \delta$. Then from (4.10) we have $|\alpha\Delta| \gg 1$. Neglecting quantities of order $\mathrm{e}^{-\alpha\Delta}$, we see from (4.11) that

$$A \doteqdot -B \doteqdot \frac{c\Delta}{\alpha\Delta - 2} \doteqdot \frac{c}{\alpha}. \tag{5.1}$$

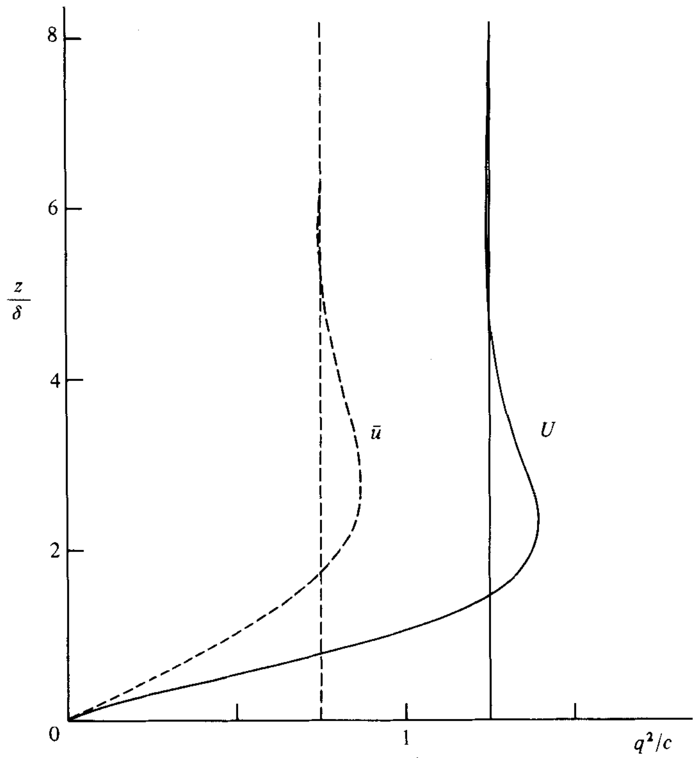


FIGURE 2. The streaming velocity $U$ and the Eulerian-mean velocity $\bar{u}$ in the boundary-layer at the bottom when $\varpi = 0$ and $\Delta/\delta \gg 1$.

Hence near the bottom, when $\alpha z = O(1)$, we find

$$\psi_1 = A(e^{-\alpha z} + \alpha z - 1)\, e^{i(kx-\sigma t)}. \tag{5.2}$$

This is the stream function for a Stokes layer (see Lamb 1932, §347) in which as $\alpha z \to \infty$ the horizontal velocity $\epsilon\psi_{1z}$ tends to $(bc/\Delta)\, e^{i(kx-\sigma t)}$, as in §2 above. Similarly near the upper surface, setting $z = \Delta + z'$, with $\alpha z' = O(1)$, we find, after approximating $A$ and $B$ more closely, that

$$\psi_1 = A(\alpha\Delta - 1 + \alpha z' - e^{\alpha z'})\, e^{i(kx-\sigma t)}, \tag{5.3}$$

which represents a similar Stokes layer on the underside of the membrane ($z' < 0$), but with an imposed vertical velocity $i\sigma b\; e^{i(kx-\sigma t)}$ as in §2.

It will be noticed that the two Stokes layers are nearly but not quite symmetrical with respect to the mean level, the strength of the bottom layer being greater than that at the upper membrane by a factor $|1 - 2/\alpha\Delta|^{-1}$, with a slight phase difference of order $\delta/\Delta$.

The mass-transport velocity $U$ in the lower layer is easily calculated from (4.28) together with (5.2). When $\varpi = 0$ we obtain

$$U = \frac{q^2}{4c}\left(5 - 8\, e^{-z/\delta} \cos\frac{z}{\delta} + 3\, e^{-2z/\delta}\right), \tag{5.4}$$

where $q = bc/\Delta$, the amplitude of the first-order horizontal oscillatory velocity just outside the boundary layer. At this point the streaming velocity is $U = 5q^2/4c$ just as in (1.1). The velocity profile within the layer is shown in figure 2, plotted against $z/\Delta$. We show also the profile of $\bar{u} = \epsilon^2\bar{\psi}_{2z}$, the mean velocity at a fixed point. As will be seen, this tends to a different value, namely $3q^2/4c$, and is quite distinct from $U$.

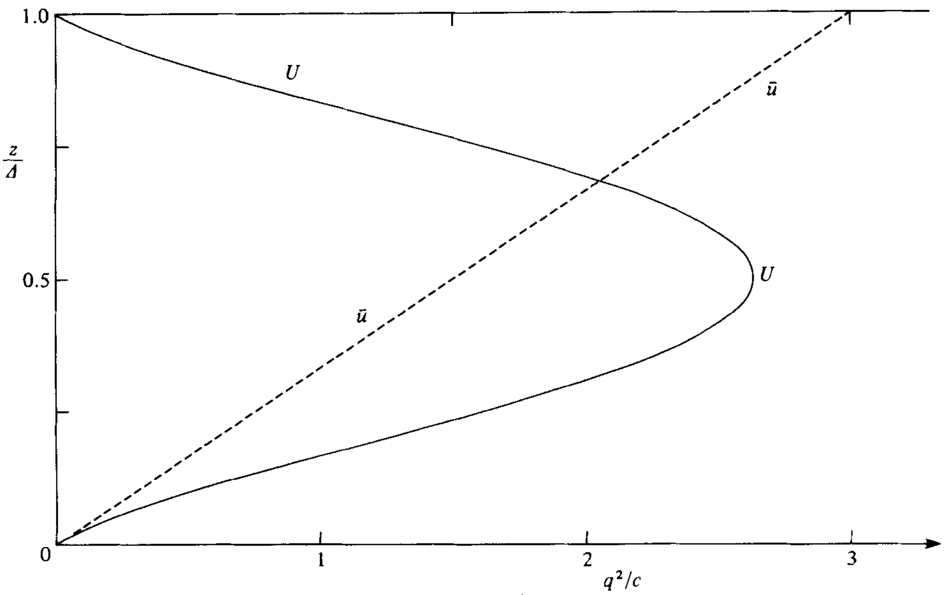


FIGURE 3. The streaming velocity $U$ and the Eulerian-mean velocity $\bar{u}$ as functions of $z/\Delta$, when $\varpi = 0$ and $\Delta/\delta \ll 1$.

The streaming in the upper boundary-layer, which we get from (4.26) and (5.3), is entirely similar to (5.4), being given by reflecting (5.4) in the mean level $z = \frac{1}{2}\Delta$.

Thus we have shown that, in the case when $\Delta \gg \delta$, there is no significant interaction between the boundary-layers, and the mean flow is as described in §2.

In the opposite case when $\Delta \ll \delta$, the first-order solution (4.9) reduces to

$$\psi_1 = c\Delta\left(\frac{3z^2}{\Delta^2} - \frac{2z^3}{\Delta}\right)e^{i(kx-\sigma t)}, \tag{5.5}$$

so

$$\epsilon\psi_{1z} = 6q\,\zeta(1-\zeta)\,e^{i(kx-\sigma t)}, \tag{5.6}$$

where $\zeta = z/\Delta$ and from (4.28)

$$U = \frac{3q^2}{c}[\zeta(1-\zeta) + 10\zeta^2(1-\zeta)^2] - \frac{1}{2}\frac{\varpi\Delta^2}{\rho\nu}\zeta(1-\zeta). \tag{5.7}$$

This represents a quartic velocity profile (see figure 3) with a total transport

$$M = \frac{3}{2}\frac{q^2\Delta}{c} - \frac{1}{12}\frac{\varpi\Delta^3}{\rho\nu}. \tag{5.8}$$

If the flow is blocked so that $M = 0$, the resulting pressure-gradient $\varpi_{\max}$ is given by

$$\varpi_{\max} = 18\frac{\rho\nu q^2}{c\Delta^2}. \tag{5.9}$$

For general values of the ratio $\Delta/\delta$ it is clear that the profile of the mass-transport velocity has the form

$$U = \frac{q^2}{c}\Phi\left(\zeta, \frac{\Delta}{\delta}\right) - \frac{\varpi\Delta^2}{2\rho\nu}\zeta(1-\zeta), \tag{5.10}$$

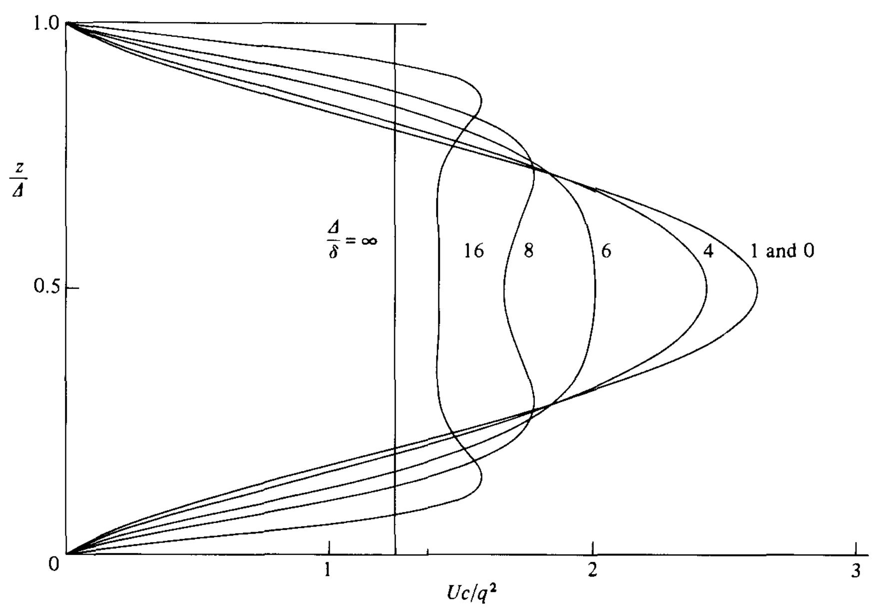


FIGURE 4. Profiles of the streaming velocity $U$ when $\varpi = 0$ and $\Delta/\delta = 1, 4, 6, 8, 16$ and $\infty$.

where $q$ is the typical horizontal velocity $bc/\Delta$ (cf. (2.7)). If we define a Reynolds number $R$ by

$$R = \frac{q\Delta}{\nu} = \frac{bc}{\nu} \tag{5.11}$$

then $\Delta/\delta$ is related to $R$ by

$$\frac{\Delta}{\delta} = \frac{k\Delta R^{\frac{1}{2}}}{(2bk)^{\frac{1}{2}}}. \tag{5.12}$$

In figure 4 we have plotted the function $\Phi$ for a number of different values of $\Delta/\delta$. The transition from the profile (5.7) at low values of $\Delta/\delta$ to the double boundary-layer profile (5.4) at high values of $\Delta/\delta$ can be clearly seen. Between $\Delta/\delta = 1$ and 6 the velocity profile is unimodal. Between $\Delta/\delta = 6$ and 8 the curvature at the central level $z/\Delta = 0.5$ changes sign and the profile becomes bimodal. As $\Delta/\delta$ increases further, the velocity maxima move apart towards the boundaries and new, less pronounced, oscillations develop near the centre. Finally as $\Delta/\delta \to \infty$ the profile tends to the limiting form indicated by the straight lines. This is the limiting form used in §2.

It can be seen that even when $\Delta/\delta$ is as low as 4 there is a strong interaction between the two boundary layers. Remarkably, however, little change in the profile takes place when $\Delta/\delta < 4$. Indeed the profile for $\Delta/\delta = 1$ is practically indistinguishable from the limiting form (5.8) corresponding to $\Delta/\delta = 0$.

In figure 4 we have plotted only the profiles corresponding to zero pressure gradient, $\varpi = 0$. For general values of $\varpi$ one has only to add a parabolic velocity profile, as in (4.28).

The profiles for $U$ are all symmetric about the mean level, in spite of the asymmetry in the first-order velocity $\psi_{1z}$. The boundary conditions (4.7) and (4.8) for $\psi_1$ are indeed asymmetric, but could be made symmetric by the addition of a small transverse (vertical) velocity $-\frac{1}{2}b\sigma\,e^{i(kx-\sigma t)}$, independent of $z$. This would not affect the mechanics of the drift velocity $U$ at lowest order in $\epsilon$ and $k\Delta$. Hence, provided that the amplitude $b$ of the vertical displacement is small compared with the thickness

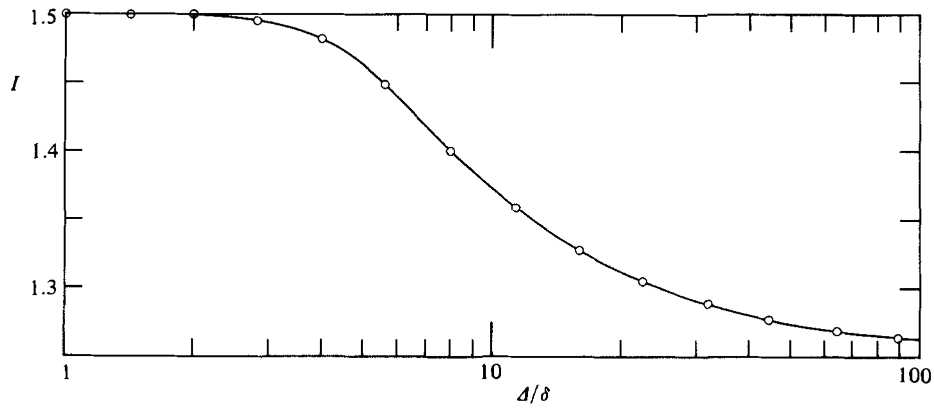


FIGURE 5. The integral $I$, giving the total mass flux in the fluid layer at zero pressure-gradient (see (5.13)).

$\Delta$, and also that $\Delta$ is small compared with the wavelength, the mass-transport velocity must be symmetric.

From equation (5.10) the total volume flux $M$ can be expressed in the form

$$M = \frac{q^2\Delta}{c} I\left(\frac{\Delta}{\delta}\right) - \frac{1}{12}\frac{\varpi\Delta^3}{\rho\nu}, \tag{5.13}$$

where

$$I = \int_0^1 \Phi \, \mathrm{d}\zeta. \tag{5.14}$$

In figure 5 we have plotted $I$ as a function of $\Delta/\delta$. Evidently $I$ is almost a constant, lying always between the two limiting values 1.5 and 1.25.

From (5.13) the maximum pressure gradient $\varpi_{\max}$ is given by

$$\varpi_{\max} = 12\frac{\nu q^2}{\Delta^2 c} I, \tag{5.15}$$

or, if we express $q$ in terms of the vertical displacement $b$ at the upper boundary ($q = bc/\Delta$), then

$$\varpi_{\max} = 12\frac{\nu b^2 c}{\Delta^4} I. \tag{5.16}$$

This shows that, given the measured parameters $b$, $c$ and $\Delta$, the pressure gradient depends primarily upon $\nu$, at least for small values of $b/\Delta$. Thus the replacement of the laminar coefficient $\nu$ by an effective eddy coefficient $\nu_e$ might account for the observed pressure difference $[p]$.

## 6. Effects of finite amplitude

When $b/\Delta$ is no longer small, the formulae of §5 must be modified. A general solution will not be attempted here, but we note that in the extreme case $b = \Delta$, when the whole layer is occluded by the perturbation, then $q = bc/\Delta = c$. Since the fluid is carried along with mean velocity $c$ and the mean thickness of the layer is still equal to $\Delta$, the transport $M$ is

$$M = c\Delta = \frac{q^2\Delta}{c}. \tag{6.1}$$

In other words (5.13), with $\varpi = 0$, remains valid if we substitute $I = 1.00$. This represents a change of less than 50% from the value for $b/\Delta \ll 1$.

On the other hand the pressure gradient $\varpi$ must be more drastically affected by the finite amplitude. For in the constricted portions of the channel, where the fluid flow tends to be reversed, not only is the strength of the oscillating component of the current greater (at zero mean flux) owing to conservation of mass, but also the pressure gradient is more than proportionately increased. In the limiting case $b = \Delta$ it is clear that $\varpi_{\max}$ must tend to infinity. Hence (5.15) and (5.16) must be serious underestimates, even at some values of $b/\Delta$ less than 1.

## 7. End-effects

We have so far neglected any effects due to the two ends of the bag. But if, for example, the flow in the pipe is blocked, the condition of zero net flow at the down-wave end may result in a reflected, damped elastic wave, which will tend to increase the amplitude $p_1$ of the pressure fluctuation there; in fact $p_1$ may be about double the corresponding amplitude far from the ends, i.e.

$$p_1 \sim 2\rho g a \operatorname{sech} kh, \tag{7.1}$$

and similarly for the fluctuation $p_2$ at the other end. The pressure difference between the two ends will be at most

$$p_1 + p_2 \sim 4\rho g a \operatorname{sech} kh \tag{7.2}$$

when the phases are opposite. This estimate agrees quite well with the relation between the peak-to-peak pressure $P$ and the wave height $H$ in figure 6 of Allison (1982). At the same time, the observed irregularities in the curve for $P$ may be accounted for by phase differences between the two ends of the bag. The effect of the two ends on the *second*-order mean-pressure difference is, however, more difficult to estimate.

## 8. Conclusions

We have calculated the mean-drift velocities induced by water waves progressing over a thin flexible bag laid on the bed of a wave tank. The induced velocities are similar to those measured by Allison, but the calculated pressure gradients are smaller than those observed.

Among the assumptions in our calculation are that the flow was laminar, and that the amplitude of the vertical displacement $b$ of the bag was small compared with the thickness $\Delta$ of the contained layer of fluid. Part of the discrepancy may be due to turbulence in the fluid. Although turbulence can be partly represented by an eddy viscosity $N(\bar{z})$, it is no longer true, in the presence of interaction between the boundary layers, that the streaming is independent of $N$.

A more probable cause for the discrepancies is the finite value of $b/\Delta$. This ratio was assumed constant over the surface of the bag. But any values much greater than the assumed value of 0.27 would certainly have the effect of increasing the pressure gradient, without drastically altering the total mean flow.

For small displacements, the general method of calculation given in §4 above (which was developed previously for water waves) provides a very convenient framework for problems of this kind. In interpreting the theoretical solutions it is essential to distinguish between the Eulerian- and Lagrangian-mean velocities, a distinction not always observed in previous studies of peristaltic flow. An example is given in the Appendix.

## Appendix. On the time-average velocity $\bar{u}$

As pointed out in §§2 and 4, the time-average $\bar{u}$ of the velocity at a fixed point differs essentially from the particle drift velocity $U$. In fact from (4.16) we have in general

$$\bar{u} = U - \frac{1}{2c}(\overline{\psi_1^2})_{zz} \tag{A 1}$$

$$= U - \frac{1}{c}(\overline{\psi_1 \psi_{1zz}} + \overline{\psi_{1z}^2}). \tag{A 2}$$

In figure 6 we have plotted $\bar{u}$ for comparison with $U$ in dimensionless form when $\Delta/\delta = \sqrt{14}$. As will be seen, not only is $\bar{u}$ quite different from $U$, but it is also asymmetrical. Whereas $U$ must vanish on both upper and lower boundary, $\bar{u}$ need vanish only on the lower boundary $z = 0$.

The point is relevant to some previous discussions of peristaltic flow, for example Fung & Yih (1968) and Jaffrin & Shapiro (1971). Thus Fung & Yih studied the two-dimensional flow induced by small oscillations of the two boundaries of a thin fluid layer. Their analysis is directly comparable with ours, with a Reynolds number $R'$ equivalent to our $(\Delta/\delta)^2/k\Delta$. In numerical examples they took $\frac{1}{2}k\Delta R'$ equal to 0.1, 0.4 and 7.0. However, their discussion is solely in terms of the second-order mean velocity $\bar{u}$, not the drift velocity $U$. The appropriate condition for a reflux, or reverse flow, is surely not $\bar{u} < 0$ but rather $U < 0$. Moreover, with a moving boundary the total flux $M$ cannot be obtained by integrating $\bar{u}$ up to the mean boundary; additional second-order terms are involved. Lastly, in Fung & Yih's paper it is not clear that the correct boundary condition $U = 0$ has been employed.

The correct solution to Fung & Yih's problem can in fact be written down immediately from the analysis of §4 above. Moving the origin $O$ to the central level

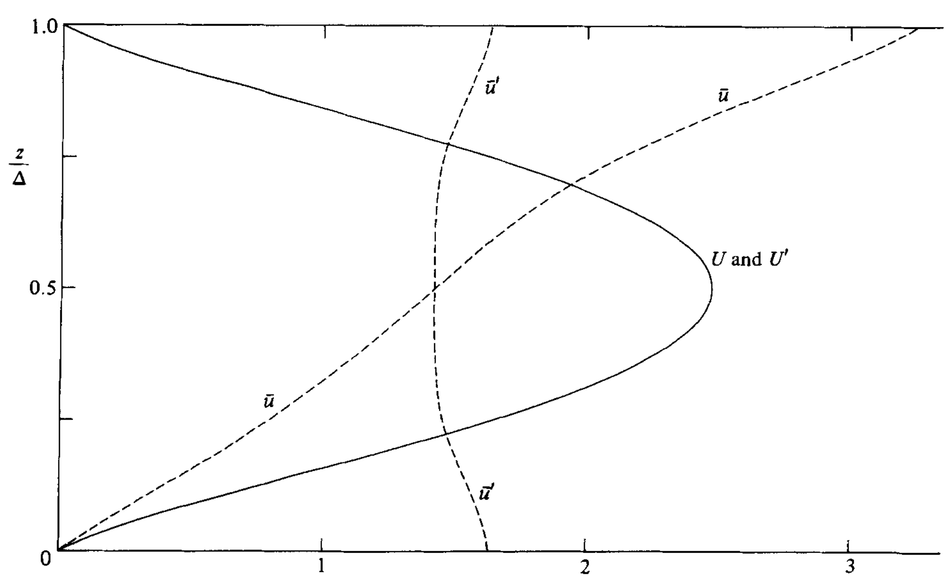


FIGURE 6. Comparison of the Eulerian- and Lagrangian-mean velocities ($\bar{u}$ and $U$) when $\Delta/\delta = \sqrt{14} = 3.742$. Also shown is the Eulerian-mean velocity $\bar{u}'$ in the problem of Fung & Yih (1968).

$z = \frac{1}{2}\Delta$ by writing $z' = z - \frac{1}{2}\Delta$ we have for the first-order motion $\psi_1'$ the boundary conditions

$$\left.\begin{aligned} \psi_1' &= \tfrac{1}{2}c\Delta\, e^{i(kx-\sigma t)}, \quad \psi_{1z} = 0 \quad \text{when} \quad z' = \tfrac{1}{2}\Delta, \\ \psi_1' &= -\tfrac{1}{2}c\Delta\, e^{i(kx-\sigma t)}, \quad \psi_{1z} = 0 \quad \text{when} \quad z' = -\tfrac{1}{2}\Delta, \end{aligned}\right\} \tag{A 3}$$

where $\epsilon = b/\Delta$ and the amplitude of the vertical displacement at each membrane equsls $\frac{1}{2}b$. The required solution is

$$\psi_1' = \frac{\alpha z' \cosh(\frac{1}{2}\alpha\Delta) - \sinh \alpha z'}{\frac{1}{2}\alpha\Delta \cosh(\frac{1}{2}\alpha\Delta) - \sinh(\frac{1}{2}\alpha\Delta)} \frac{c\Delta}{2} e^{i(kx-\sigma t)}. \tag{A 4}$$

This compares with the first-order solution $\psi_1$ of (4.9), which can be expressed simply as

$$\psi_1 = \psi_1' + \tfrac{1}{2}c\Delta\, e^{i(kx-\sigma t)}. \tag{A 5}$$

Clearly $\psi_{1z}' = \psi_{1z}$ and in (4.21) $Q' = Q$. Therefore by (4.26) the mass-transport velocity $U'$ is given by

$$U' = U, \tag{A 6}$$

and so is described by the curves in figures 4–6.

The corresponding Eulerian-mean velocity $\bar{u}'$ can be found from the relation

$$\bar{u}' = U' - \frac{\epsilon^2}{c}(\overline{\psi_1' \psi_{1zz}} + \overline{\psi_{1z}'^2}), \tag{A 7}$$

which in view of (A 5) reduces to

$$\bar{u}' = \bar{u} + \tfrac{1}{2}\epsilon^2 \Delta\, \overline{e^{i(kx-\sigma t)} \psi_{1zz}}. \tag{A 8}$$

Numerical evaluation of this expression yields the curve for $\bar{u}'$ in figure 6, which agrees closely with figure 3(*c*) of Fung & Yih (1968).

The fact that the Eulerian-mean velocities $\bar{u}$ and $\bar{u}'$ in the two problems are quite different, while the Lagrangian-mean velocities $U$ and $U'$ are equal confirms that the latter have a greater physical significance.

The author is indebted to Dr H. Allison for stimulating correspondence and to Drs T. J. Pedley and S. J. Hogan for comments.

## REFERENCES


ALLISON, H. 1983 Streaming of fluid under a near-bottom membrane for utilization of sea-wave energy. *J. Fluid Mech.* **137**, 385–392.

BAGNOLD, R. A. 1947 Sand movement by waves: some small-scale experiments with sand of very low density. *J. Inst. Civ. Engrs Lond.* **27**, 447–469.

BARTON, C. & RAYNOR, S. 1968 Peristaltic flow in tubes. *Bull. Math. Biophys.* **30**, 663–683.

FUNG, Y. C. & YIH, C. S. 1968 Peristaltic transport. *Trans. ASME* E: *J. Appl. Mech.* **35**, 669–675.

JAFFRIN, M. Y. & SHAPIRO, A. H. 1971 Peristaltic pumping. *Ann. Rev. Fluid Mech.* **3**, 13–36.

LAMB, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

LONGUET-HIGGINS, M. S. 1953 Mass transport in water waves. *Phil. Trans. R. Soc. Lond.* A **245**, 535–581.

LONGUET-HIGGINS, M. S. 1956 The mechanics of the boundary-layer near the bottom in a progressive wave. In *Proc. 6th Intl Conf. on Coastal Engng*, C. 10, pp. 184–193.

RAYLEIGH, LORD 1884 On the circulation of air observed in Kundt's tubes, and on some allied acoustical problems. *Phil. Trans. R. Soc. Lond.* **175**, 1–21.

RUSSELL, R. C. H. & OSORIO, J. D. C. 1956 An experimental investigation of drift profiles in a closed channel. In *Proc. 6th Intl Conf. on Coastal Engng*, C. 10, pp. 171–183.